AD-A071 100 MASSACHUSETTS INST OF TECH CAMBRIDGE LAB FOR INFORMA--ETC F/G 20/10
FILTERING THEORY AND QUANTUM FIELDS.(U)
APR 79 S K MITTER AFOSR-77-3281
UNCLASSIFIED LIDS-P-902 AFOSR-TR-79-0812 NL

| OF |
AD
A071100

END
DATE
FILMED
8-79
DDC

2

FILTERING THEORY AND QUANTUM FIELDS[1]

by

Sanjoy K. Mitter
Department of Electrical Engineering and Computer Science
and
Laboratory for Information and Decision Systems
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

(1) This research has been supported by the Air Force Office of Scientific Research under grant AFOSR-77-3281. This paper was presented in the Conference on Algebraic and Geometric Methods in Systems Theory held at Bordeaux, France, September 1979.

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH (AFSC)
NOTICE OF TRANSMITTAL TO DDC
This technical report has been reviewed and is approved for public release IAW AFR 190-12 (7b).
Distribution is unlimited.
A. D. BLOSE
Technical Information Officer

I. Introduction.

In this paper we describe certain remarkable connections that exist between mathematical developments in quantum field theory (and euclidean field theory) and filtering theory (and in general system theory). Roughly speaking, the Kalman filter is the mathematical analog of a free quantum field (in a precise sense) and the study of non-linear filtering is the analog of the study of interacting quantum fields. Due to lack of space we only sketch this theory in this paper and the details of this work will be presented elsewhere. Some of these ideas were presented in an earlier paper of this author (MITTER [1]).

II. The Free Quantum Field (after I.E. Segal).

We shall assume that the reader is familiar with the theory of gaussian measures on infinite dimensional spaces as developed by Segal and Gross (cf. SEGAL [1] and GROSS [1]).

Let H' be a given real Hilbert space and let g denote the weak probability distribution on H' known as the centred isotropic normal distribution. This may be defined as follows:

A cylinder set in H' can be described in

$$C = \{x \in H' | (\langle x, y_1\rangle, \ldots, \langle x, y_n\rangle) \in A\}$$

where $y_1, \ldots, y_n$ are orthonormal, $A \in \mathcal{B}(R^n)$ and $\langle\cdot,\cdot\rangle$ denotes the scalar product on H'. Let $P : H' \to \text{span}(y_1, \ldots, y_n)$ denote the orthogonal projection. Then

$$C = \{x \in H' | Px \in D\} \quad \text{where } D = C \cap \text{Range}(P)$$

For each $\sigma > 0$, let

$$(2.1) \quad \mu_\sigma(C) = (2\pi\sigma)^{-n} \int_D e^{\frac{-||x||^2}{2\sigma}} dx$$

where n = dim (Range (P)) and dx = Lebesque measure on Range (P). (The centred isotropic normal distribution corresponds to taking $\sigma = 1$).

The measure as defined above is countably additive on the σ-ring S_K (collection of cylinder sets based on K = span $(y_1, \ldots, y_n)$), but is only finitely additive on R, the ring of all cylinder sets. Also μ as defined above does not have a countably additive extension on H'.

An integration theory over (H',g) can be developed and the space $L^2(H',g)$ can be constructed using appropriate completion arguments (cf. SEGAL [2] for a survey of these ideas).

Definition: The Free Boson field over a given complex Hilbert space H may be defined as $\Phi(H) = (K, W, \Gamma, v)$ consisting of

(1) a complex Hilbert space K

(2) a Weyl system W on H, that is, a strongly continuous mapping $z \mapsto W(z)$: $H \to \mathcal{U}(K)$ (the space of unitary operators on K) satisfying the Weyl relations

(2.2) $W(z)W(z') = \exp(\frac{i}{2} \mathrm{Im}\langle z,z'\rangle)W(z+z'), \forall z, z' \in H$

(3) A continuous representation $\Gamma : \mathcal{U}(H) \to \mathcal{U}(K)$

(2.3) $\Gamma(U)W(z)\,\Gamma(U)^{-1} = W(Uz)\ \forall U \in \mathcal{U}(H)\ \forall z \in H$

(4) a unit vector $v \in K$ s.t. $\Gamma(U)v = v$ and v is a cyclic vector for $\mathcal{W}$

(5) Γ is positive in the sense that if $A \in \mathcal{L}(H)$, $A \geq 0$ and self-adjoint, then $d\Gamma(A)$ = self-adjoint generator of the one-parameter unitary group.

$[\Gamma(e^{itA})|t \in R]$ is also non-negative. ▨

Using appropriate standardizations SEGAL [3] has proved

Theorem 1. There exists a unique Weyl system W and a representation Γ of U(H) such that $(L^2(H',g), \mathcal{W}, \Gamma, v)$ represents the free Boson field.

The generator $d\Gamma(A)$ as defined previously is the "number operator" of quantum mechanics. It is best studied not on $L^2(H',g)$ but on the isomorphic Fock space $e^H = H^o \oplus H \oplus H \oplus_S H \oplus \ldots$ considered as a direct sum of tensor products of Hilbert spaces. Here $H^o = \mathbb{C}$ and H is the complexification of H'. The action of $d\Gamma(A)$ on e^H is essentially the action of $A \oplus \ldots \oplus_S A$ on the Hermite polynomials, which span e^H. If we take $A = I$ then $d\Gamma(I)$ essentially represents the dynamics of an infinite number of harmonic oscillators.

Now the free Boson field could also be interpreted as a unitary representation of the Heisenberg group and this interpretation will be important in the sequel.

Before we close this section we shall state a theorem of Shale (SHALE [1]).

We have now constructed the free field starting with a Guassian measure on H' with covariance $||x||^2$. Let $A : H' \to H'$ be a bounded positive operator with bounded inverse and let g' denote Gauss measure on H' with covariance $\frac{1}{2}||Ax||^2$. We can now construct the corresponding quantum field where the representation space K is $L^2(H',g')$. We can ask the question whether the two fields are unitary equivalent. This is really a question on relative absolute continuity of measures and the answer is provided completely:

Theorem 2. A necessary and sufficient condition for the two fields to be unitary equivalent are that $A - I$ be Hilbert Schmidt. ▨

III. Kalman Filtering and the Free Quantum Field.

Let (Ω, A, P) be the probability space on which all random variables are defined. Consider the linear filtering problem

$$(3.1)\quad \begin{cases} y_t = \int_o^t z_s ds + \eta_t \\ z_s = (h, x_s) \\ x_t = \int_o^t Fx_s ds + \int_o^t g dW_s \end{cases}$$

In the above, y is the scalar observation process, z is the scalar signal process, x is an R^n-valued state process, F is an n x n matrix and g is an n-vector. η and W are independent standard Wiener processes.

We first consider the linear filtering problem of recursively obtaining $\hat{x}_t \triangleq E(x_t | F_t^y)$ where F_t^y is the σ-field generated by $\{y_s, 0 \leq s \leq t\}$. As is well known the innovation process

$$(3.2) \quad \nu_t = y_t - \int_0^t (h, \hat{x}_s) ds$$

is standard Brownian motion. Now, from standard theory

$$\hat{x}_s = \int_0^s k_s(\tau) dy_\tau \ , \text{ where } k_s(\cdot) \in L^2(0, s; R^n).$$

Hence equation (3.2) can be written as

$$(3.3) \quad \nu_t = y_t - \int_0^t (h, \int_0^s k_s(\tau) dy_\tau) ds,$$ which we write in obvious operator form as

$$(3.4) \quad \nu = (I - K) y,$$ where we regard y as an element of $H^1(0, 1; \mu_y)$ and ν as an element $H^1(0, 1; \mu_\nu)$ (H^1 is the space of absolutely continuous functions with derivatives in L^2 and μ_y is the Gauss measure corresponding to the y-process and μ_ν is the Gauss measure corresponding to the ν process). The operator K is obviously Hilbert-Schmidt. The Kalman filter is essentially the action of the operator I - K on the observation process y to produce the innovation process ν. By the Segal-Feldman theorem μ_y and μ_ν are equivalent and by Theorem 2 (Shale) the quantum field's corresponding to the Gaussian measures μ_y and μ_ν are unitarily equivalent. Hence the Kalman filter is isomorphic to the free quantum field obtained from the innovations process ν.

<u>Remark</u>: In terms of the notation of section 2, from Stone's theorem $W(z) = e^{i\psi(z)}$ where ψ is a mapping from H to the self-adjoint operators on K. Now if H_r is any real subspace of H such that $H = H_r \oplus iH_r$ (over the real field) then the restriction map $\psi|_{H_r}$, relative to the expectation values defined by the functional $E(A) = \langle Av,v\rangle$ for any operator A in the ring of operators generated by $e^{i\psi(x)}$, $x \varepsilon H_r$ is within a certain isomorphism the centred isotropic normal process over H_r. This may be seen as follows. The $\psi(x)$ for $x \varepsilon H_r$ are mutually commutative and determine a maximal abelian ring of operators and hence may be identified with the multiplication operators associated with certain real measurable functions acting on $L^2(M,m)$ for a suitable measure space (M,m). Now $E(e^{i\psi(x)}) = e^{-\frac{1}{4}||x||^2} = E\left(e^{i\phi(x)/2^{\frac{1}{2}}}\right)$ where ϕ is the weak-distribution corresponding to white noise (centred isonormal process). Hence Ω may be chosen as the probability space of white noise such that expectation values for operators in K and for random variables on Ω are equal.

IV. <u>Other Applications</u>.

The isomorphism exhibited between Kalman filtering and the free quantum field has other applications.

Fock space ideas have applications to certain non-linear filtering problems (cf. MARCUS-MITTER-OCONE).

Suppose $A : H' \to H'$ is a contraction. Then $\Gamma(A)$ is a contraction from $L^q(H')$ to $L^p(H')$ for $1 \leq q \leq p \leq \infty$ provided $||A|| \leq \sqrt{\frac{q-1}{p-1}}$.

This is a theorem of Nelson (NELSON [1]). This theorem has applications to obtain estimates on the product of Ito multiple integrals and to expansion of non-linear functionals of white noise. There are also connections of these ideas to recent work of Arveson (ARVESON [1]) and together provide the framework for realization

theory and universal models for certain non-linear operators.

There are also connections to the study of the unnormalized conditional density equations for the filtering problem using Lie algebra ideas. These ideas have been investigated by Brockett (BROCKETT [1]) and the author. The fact that the free field is a unitary representation of the Heisenberg group is important in this context.

REFERENCES

W. ARVESON [1] A Spectral Theorem for Nonlinear Operators, Bull. Am. Math. Soc., 82, (1976), 511-513.

R.W. BROCKETT [1] Remarks on Finite Dimensional Nonlinear Filtering, this proceedings.

L. GROSS [1] Abstract Wiener Spaces, Proc. 5th Berkeley Symposium II, 1965, 31-42.

S.I. MARCUS, S.K. MITTER, AND D. OCONE [1] Finite Dimensional Nonlinear Estimation in Continuous and Discrete Time, presented at the International Conference on Analysis and Optimization of Stochastic Systems, Oxford, September 6-8, 1978, M.I.T., LIDS-P-855, October 1978, pp. 1-19.

S.K. MITTER [1] Modelling for Stochastic Systems and Quantum Fields, Proceedings of the 1979 IEEE Decision and Control Conference, San Diego, California, January 1979.

E. NELSON [1] The Free Markov Field, J. Func. Anal. 12 (1973), 211-227.

I.E. SEGAL [1] Tensor Algebras over Hilbert Spaces, I. Trans. Am. Math. Soc. 81 (1956), 106-134.

__________ [2] Algebraic Integration Theory, Bull. Am. Math. Soc., 71, (1965), 419-489.

__________ [3] The Complex Wave Representation of the Free Boson Field, to appear in Essays in Honour of M.G. Krein, ed. Gohberg and Kac., Academic Press.

D. SHALE [1] Linear Symmetries of the Free Boson Field, Trans. Am. Math. Soc., 103 (1962), 149-167.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER: AFOSR-TR-79-0812	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): FILTERING THEORY AND QUANTUM FIELDS (1)		5. TYPE OF REPORT & PERIOD COVERED: Interim rept.
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Sanjoy K. Mitter		8. CONTRACT OR GRANT NUMBER(s): AFOSR-77-3281
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Massachusetts Institute of technology, Dept of Electrical Engineering, Cambridge, Massachusetts 02139		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F 2304/A1
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Research/NM, Bolling AFB, Washington, D.C. 20332		12. REPORT DATE: April 1979
		13. NUMBER OF PAGES: 7
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): 8p.		15. SECURITY CLASS. (of this report): UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited. LIDS-P-942

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper we describe certain remarkable connections that exist between mathematical developments in quantum field theory (and euclidean field theory) and filtering theroy (and in general system theory). Roughly speaking, the Kalman filter is the mathematical analog of a free quantum field (in a precise sense) and the study of non-linear filtering is the analog of the study of interacting quantum fields. Due to lack of space we only sketch this theory in this paper and the details of this work will be presented elsewhere.

DD FORM 1 JAN 73 1473

410 950